Mass Transfer to a Plane below a Rotating Disk at High Schmidt Numbers

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The Lighthill transformation is used to determine the rate of mass transfer at high Schmidt numbers to a disk imbedded in a stationary plane below a rotating disk. This diffusion-layer solution breaks down near the center and leading edge of the mass-transfer disk. Solutions in these regions are reported, and a uniformly valid composite solution for the local Nusselt number is determined along with an expression for the average mass-transfer rate to the disk.

SCOPE

Exact solutions to the Navier-Stokes equations in rotating systems are often possible and permit the subsequent theoretical treatment of heat or mass transfer. The high-Schmidt-number analysis of mass transfer from axisymmetric bodies is of special interest in transport phenomena since Schmidt numbers for many chemical and electrochemical systems are large and a diffusion-layer solution to the equation of convection diffusion is possible.

The system under investigation here consists of a fluid driven by a disk which is rotating above a stationary plane. Mass transfer is allowed to take place on a coaxial disk imbedded in the plane. Mellor, Chapple, and Stokes (1968) have obtained an exact solution to the equations of motion for this problem. A closely related problem is that studied by Smith and Colton (1972) in which the fluid rotates as a solid body at a large distance from a stationary plane with a mass-transfer section.

CONCLUSIONS AND SIGNIFICANCE

By a singular-perturbation treatment for large Péclet numbers, three regions of different mass-transfer mechanisms have been found. A diffusion-layer solution applies in a region which is bounded by elliptic regions at the center and leading edge of the mass-transfer disk. The elliptic regions have been solved previously by Newman (1969a, 1973) and are applied to this problem. A uniformly valid composite solution for the local Nusselt number was determined along with an expression for the average mass-transfer rate from the disk. Good agreement was found between the high-Schmidt-number numerical results of Smith and Colton (1972) and the results of this work. An exact solution for the surface concentration was found for the case in which the flux to the disk is uniform.

Recently Smith and Colton (1972) have treated the problem of mass transfer between a fluid in solid-body rotation and a coaxial, stationary disk. The complete equation of convective diffusion was solved for various boundary conditions on the disk at various Schmidt numbers. In a companion paper, Colton and Smith (1972) report experiments conducted on the same system. For large Schmidt numbers this problem affords an excellent situation in which to apply the Lighthill (1950) transformation to obtain a similarity solution to the convective diffusion equation. In this paper we obtain a diffusion-layer solution and examine the regions in which this solution breaks down to provide a complete representation of the mass-transfer process. We have extended the problem of Smith and Colton to a high-Schmidt number treatment of mass transfer between a fluid driven by a rotating disk and a coaxial disk imbedded in a stationary plane.

HYDRODYNAMICS

Mellor, Chapple, and Stokes (1968) have solved the equations of motion for flow between an infinite rotating disk and a stationary plane. The treatment is much the same as for the rotating disk. A von Kármán (1921) transformation reduces the problem to a set of coupled, ordinary differential equations. The radial velocity component is given by

$$v_r = r\Omega F(\zeta) \tag{1}$$

where $\zeta = y\sqrt{\Omega/\nu}$. Figure 1 shows the system. One parameter, a dimensionless separation distance Z, remains in the dimensionless problem.

$$Z = L\sqrt{\Omega/\nu} \tag{2}$$

The square of this parameter is known as the Ekman number.

For the treatment to follow we will need the radial velocity derivative $F'(0)$ evaluated at the surface of the plane. Figure 2 shows this derivative as a function of the dimensionless separation distance Z. This plot was prepared from the results of Mellor et al. (1968). For large Z, $F'(0)$ approaches the asymptotic value of -0.1622, which one would obtain for a fluid rotating as a solid body with angular velocity $\omega = \gamma\Omega$ above a stationary plane. This latter problem was first solved by Bödewadt (1940). The flow near the rotating disk has been calculated by Rogers and Lance (1960) and corresponds to the problem of a fluid in a state of solid body rotation above a rotating disk. The value of $\gamma = 0.3095$ may be determined from the ratio of the asymptotic value of $F'(0) = -0.1622$ for large Z obtained by Mellor et al. and the value $F'(0) = -0.94197$ corresponding to the Bödewadt solution. The same result may be obtained by matching the axial velocity for the Bödewadt problem with that studied by Rogers and Lance.

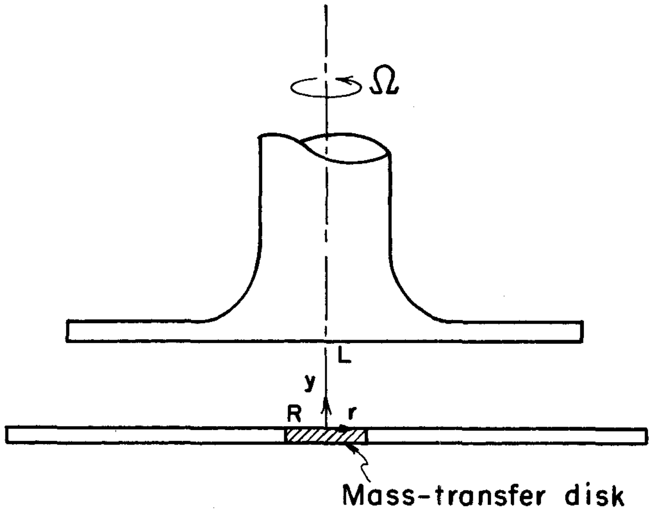

Fig. 1. Mass-transfer disk imbedded in a stationary plane below a rotating disk.

MASS TRANSFER

As shown in Figure 1, the mass-transfer system consists of a stationary disk of uniform concentration which is imbedded in a plane below a rotating disk. At high Schmidt numbers, mass transfer takes place in a thin diffusion layer near the surface of the plane. A careful analysis shows that there are two more regions (one at the center and one at the leading edge of the stationary disk) having different mass-transfer mechanisms. A sketch of these regions is shown in Figure 3.

Region 1: The Diffusion Layer

In this region, convection and diffusion normal to the disk are important, while radial diffusion is negligible. The Lighthill (1950) transformation provides a similarity solution to the convective diffusion equation, as pointed out by Acrivos (1960). Newman (1968) has shown how to apply this transformation to an axisymmetric body and more recently (1972) to the case of a rotating sphere. At high Schmidt numbers, the diffusion layer is thin compared to the hydrodynamic boundary layer. It is then valid to approximate the radial velocity in this region by the first term in an expansion in distance from the disk

$$v_r = -y\beta(r) \tag{3}$$

where $\beta = -\partial v_r/\partial y$ at $y = 0$. With this approximation and the equation of continuity, the convective diffusion equation in cylindrical coordinates is

$$-y\beta\frac{\partial\Theta}{\partial r} - \frac{1}{2}y^2\frac{(r\beta)'}{r}\frac{\partial\Theta}{\partial y} = D\left(\frac{\partial^2\Theta}{\partial y^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Theta}{\partial r}\right) \tag{4}$$

For uniform concentration on the disk and neglect of radial diffusion, the Lighthill transformation yields the solution to Equation (4)

$$\Theta_1 = \frac{1}{\Gamma(4/3)}\int_0^\eta e^{-x^3}\,dx \tag{5}$$

in terms of the Lighthill similarity variable

$$\eta = \frac{y\xi}{R}\left[\frac{2Pe}{3(1-\xi^3)}\right]^{1/3} \tag{6}$$

where $\xi = r/R$ is a dimensionless radius and $Pe = \beta R^3/2rD$ is a Péclet number. For the problem studied here, we have from Equation (1) that $\beta = -r\Omega\sqrt{\Omega/\nu}\,F'(0)$, so that $Pe = -1/2\,F'(0)\,Sc\,Re^{3/2}$ and $Re = R^2\Omega/\nu$ is the Reynolds number based on the radius of the mass-transfer disk. The function in Equation (5) is tabulated (Abramowitz et al., 1964).

The local Nusselt number is then given by

$$Nu_1 = 2R\frac{\partial\Theta_1}{\partial y}\bigg|_{y=0} = \frac{2\xi}{\Gamma(4/3)}\left[\frac{2Pe}{3(1-\xi^3)}\right]^{1/3} \tag{7}$$

Region 2: Central Region

The diffusion-layer solution would be expected to break down in a small region near the center of the disk due to the neglect of radial diffusion. In this region none of the components of convection or diffusion dominates. If stretched coordinates

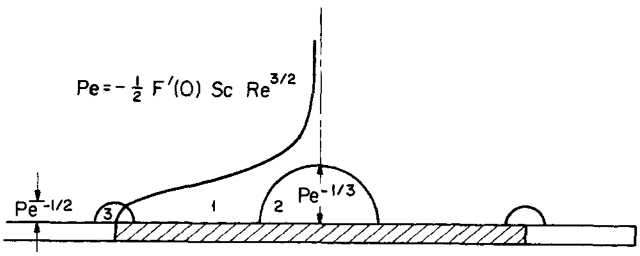

Fig. 3. Regions of different mass-transfer mechanisms on the stationary plane.

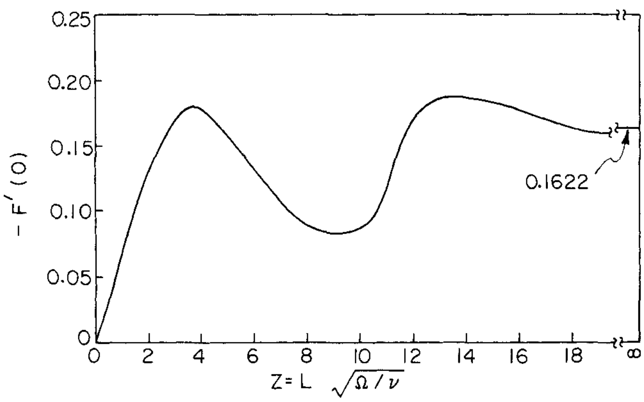

Fig. 2. Radial velocity derivative at the surface of the plane as a function of dimensionless separation distance.

$$S = \frac{r}{R}\left(\frac{4Pe}{3}\right)^{1/3} \text{ and } Y = \frac{y}{R}\left(\frac{4Pe}{3}\right)^{1/3} \quad (8)$$

are used, all the terms in the equation of convective diffusion (4) are of the same order of magnitude. With a new dimensionless concentration

$$\overline{\Theta_2} = \Theta_2\Gamma(4/3)\left(\frac{8Pe}{3}\right)^{1/3} \quad (9)$$

the resulting equation is

$$\frac{3}{2}Y^2\frac{\partial\overline{\Theta_2}}{\partial Y} - \frac{3}{2}YS\frac{\partial\overline{\Theta_2}}{\partial S} = \frac{\partial^2\overline{\Theta_2}}{\partial Y^2} + \frac{1}{S}\frac{\partial}{\partial S}\left(S\frac{\partial\overline{\Theta_2}}{\partial S}\right) \quad (10)$$

subject to the boundary conditions:

1. $\overline{\Theta_2} = 0$ at $Y = 0$, on the disk.
2. $\partial\overline{\Theta_2}/\partial S = 0$ at $S = 0$, at the axis of the system.
3. $\overline{\Theta_2} \to YS$ as $S \to \infty$, in order to match with the diffusion-layer solution.
4. As $Y \to \infty$, the term $\partial^2\overline{\Theta_2}/\partial Y^2$ should become negligible.

This elliptic equation is the same as that solved previously by Newman (1969a) for mass transfer at the rear region of a bluff, axisymmetric body at high Schmidt numbers. The local Nusselt number for this region,

$$Nu_2 = \frac{2^{2/3}}{\Gamma(4/3)}\left.\frac{\partial\overline{\Theta_2}}{\partial Y}\right|_{Y=0} = 1.778\left.\frac{\partial\overline{\Theta_2}}{\partial Y}\right|_{Y=0} \quad (11)$$

is shown in Figure 4. The value of Nu_2 at the center of the disk is 1.998, while for large S

$$\left.\frac{\partial\overline{\Theta_2}}{\partial Y}\right|_{Y=0} \to S + \frac{0.4056}{S} \text{ as } S \to \infty \quad (12)$$

Region 3: Leading-Edge Region

In this region, diffusion and radial convection dominate, while axial convection is unimportant. This elliptic region, which extends beyond the diffusion layer near the leading edge, is even smaller than the central region, see Figure 3. Leading-edge regions have been encountered in the high-Péclet-number analysis of the Graetz problem with axial diffusion and mass transfer to a plate in uniform shear flow (Newman, 1973). Substitution of the stretched coordinates

$$X = (1 - \xi)\sqrt{Pe} \text{ and } \overline{Y} = \frac{y}{R}\sqrt{Pe} \quad (13)$$

into the convective diffusion equation (4) yields, in the limit of an infinite Péclet number,

$$2\overline{Y}\frac{\partial\Theta_3}{\partial X} = \frac{\partial^2\Theta_3}{\partial X^2} + \frac{\partial^2\Theta_3}{\partial\overline{Y}^2} \quad (14)$$

subject to the boundary conditions:

1. $\Theta_3 = 0$ at $\overline{Y} = 0$, $X > 0$, on the disk.
2. $\partial\Theta_3/\partial\overline{Y} = 0$ at $Y = 0$, $X < 0$, on the plane upstream.
3. $\Theta_3 \to 1$ as $X \to -\infty$ or as $\overline{Y} \to \infty$, where concentration is at bulk value.
4. As $X \to \infty$, the $\partial^2\Theta_3/\partial X^2$ term should become negligible.

This elliptic problem has been solved by Newman (1973). Figure 5 shows the dimensionless mass-transfer rate in the leading-edge region. Near $\xi = 1$ the local Nusselt number,

$$Nu_3 = 2\sqrt{Pe}\left.\frac{\partial\Theta_3}{\partial\overline{Y}}\right|_{\overline{Y}=0} \quad (15)$$

becomes infinite like $1/\sqrt{1-\xi}$ instead of $1/(1-\xi)^{1/3}$ as predicted by the diffusion-layer solution [Equation (17)]. For large X, the mass-transfer rate matches with the diffusion-layer solution, which in terms of the variables of region 3 is

$$\left.\frac{\partial\Theta_3}{\partial\overline{Y}}\right|_{\overline{Y}=0} \to \frac{1}{\Gamma(4/3)}\left(\frac{2}{9X}\right)^{1/3} \text{ as } X \to \infty \quad (16)$$

Composite Solution

What we have obtained thus far is, in effect, the first term of a singular-pertubation expansion for the local Nusselt number. The small parameter in this treatment has been $1/Pe$. A composite solution uniformly valid over the entire mass-transfer surface may be obtained by adding the solutions for the local Nusselt number in the various regions and subtracting the common terms, namely the outer limits of the two inner regions 2 and 3. Thus,

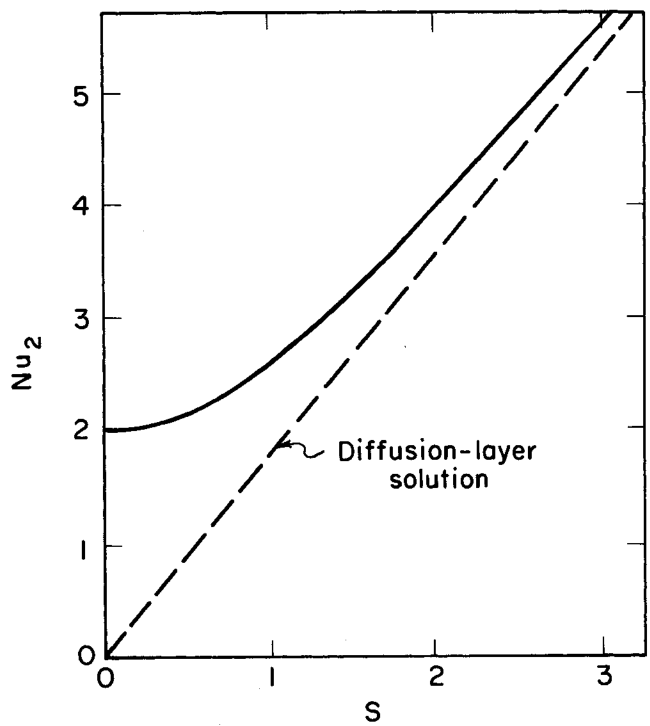

Fig. 4. Local Nusselt number in the region near the center of the mass-transfer disk.

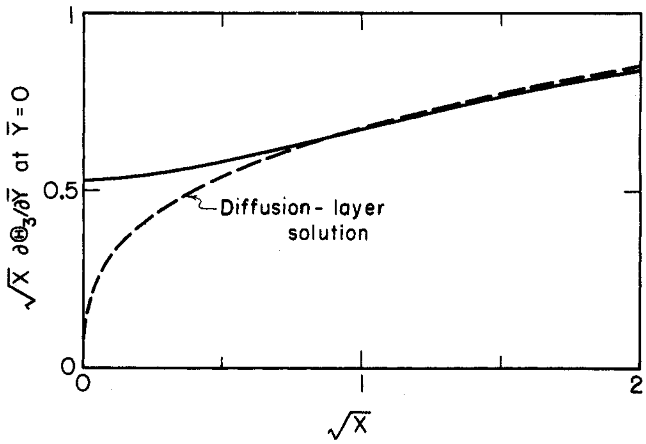

Fig. 5. Dimensionless mass-transfer rate in the leading-edge region.

$$Nu_c = Nu_1 + Nu_2 - Nu_2(S \to \infty) + Nu_3 - Nu_3(X \to \infty) \quad (17)$$

Substitution into this equation yields

$$Nu_c = \frac{2}{\Gamma(4/3)} \left\{ \xi \left[\frac{2Pe}{3(1-\xi^3)} \right]^{1/3} + \frac{1}{2^{1/3}} \left(\frac{\partial \overline{\Theta}_2}{\partial Y} \bigg|_{Y=0} - S \right) + \sqrt{Pe} \left[\Gamma(4/3) \frac{\partial \Theta_3}{\partial \overline{Y}} \bigg|_{\overline{Y}=0} - \left(\frac{2}{9X} \right)^{1/3} \right] \right\} \quad (18)$$

Figure 6 shows the composite Nusselt number for $Pe = 500$, a value chosen small enough for the central region to be seen, yet large enough so that the solution remains asymptotic. The leading-edge region is too small to be seen here.

Average Nusselt Number

The average mass-transfer rate to the disk can be expressed as

$$\overline{Nu} = 2 \int_0^1 Nu\, \xi\, d\xi = aPe^{1/3} + 0(1) \quad (19)$$

where $a = \frac{2}{\Gamma(4/3)} \left(\frac{2}{3} \right)^{1/3} = 1.9566$ comes from the diffusion layer. The neglected term of order unity comes from a correction to the diffusion-layer solution due to inclusion of the next term in the radial velocity expansion [Equation (3)]. Such an extension of the diffusion-layer solution was carried out by Acrivos and Goddard (1965) for a sphere in Stokes flow. Similar but simpler extensions are given by Newman (1969b) and by Mohr and Newman (1972). Contributions of the elliptic regions to the average Nusselt number are of even higher order, namely, $0(Pe^{-1/6})$ for the leading edge region and $0(Pe^{-2/3})$ for the central region. (These last results require consideration of the magnitude of the local Nusselt number in each region, the size of each region, and the cancellation of corrections to the diffusion-layer solution in Figure 5.)

Uniform Flux on the Disk

Smyrl and Newman (1972) have shown how to superpose the axisymmetric diffusion-layer solution to determine the surface concentration when the flux to the surface of the mass-transfer section is specified. For the case in which the flux is uniform, the exact solution

$$c_0(r) - c_\infty = - \frac{R}{\Gamma(2/3)} \left(\frac{3}{2Pe} \right)^{1/3} \frac{\partial c}{\partial y} \bigg|_{y=0} f(\theta) \quad (20)$$

is found, where

$$f(\theta) = \frac{1}{6} \ln \frac{1-\theta^3}{(1-\theta)^3} + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{1+2\theta}{\sqrt{3}} \right) - \frac{\pi}{6\sqrt{3}} \quad (21)$$

$$\theta^3 = 1 - \xi^3 \quad (22)$$

Figure 7 shows the surface concentration variation in this situation as a function of the dimensionless radius.

DISCUSSION AND CONCLUSIONS

For the case in which the mass-transfer flux to disk is uniform, the surface concentration decreases monotonically from the bulk value at the leading edge to an infinitely negative value at the center of the disk. This behavior near $\xi = 0$ is typical of mass-transfer problems with constant wall flux in the region near the rear or trailing edge of a bluff object.

Figure 6 provides a complete representation of the local mass-transfer rate to a disk which provides a surface of uniform concentration. The composite Nusselt number is seen to be infinite at the leading edge and decreases monotonically to the value 1.998 at the center of the disk. A comparison of the results obtained here may be made with the high-Schmidt-number results of Smith and Colton (1972). They report the dimensionless mass-transfer rate in terms of a Stanton-number group, which is related to the Nusselt number by

$$St\, Sc^{2/3} Re^{1/2} = \frac{Nu}{2} \left[\frac{-F'(0)}{2Pe} \right]^{1/3} \quad (23)$$

For large Schmidt numbers, only the diffusion-layer solution contributes significantly in the region away from the

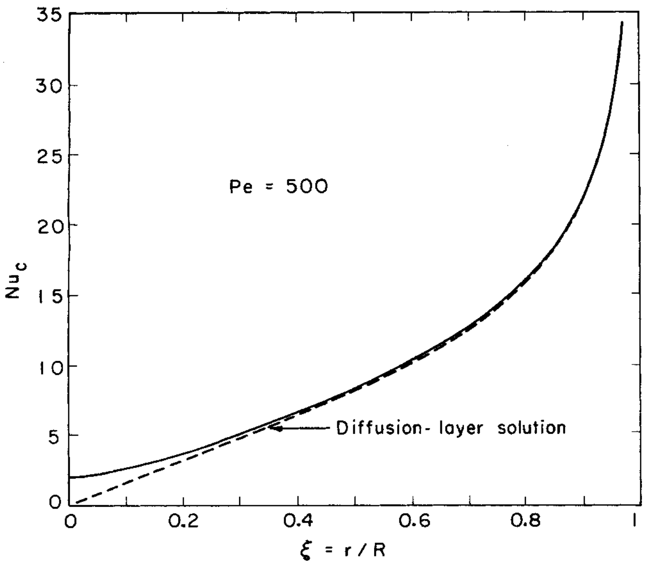

Fig. 6. Composite Nusselt number as a function of dimensionless radius, for Pe = 500.

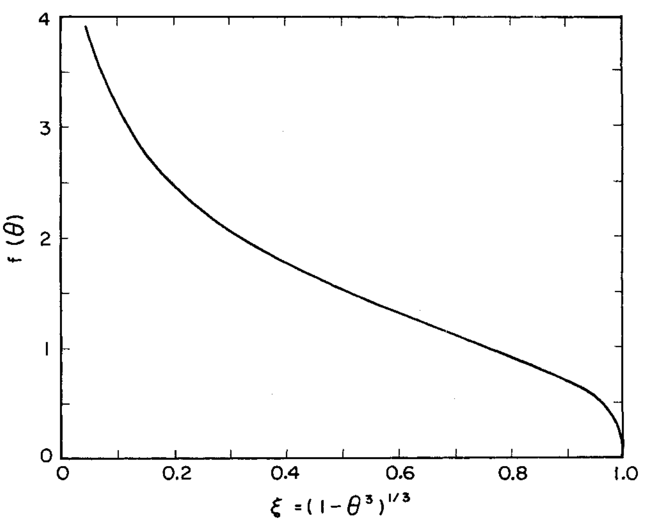

Fig. 7. Surface composition variation as a function of dimensionless radius for the condition of uniform flux to the disk. Solution for region 1 for large Schmidt numbers.

center and edge. Thus, from Equation (7) we have

$$St_1\, Sc^{2/3}\, Re^{1/2} = \frac{\xi}{\Gamma(4/3)} \left[\frac{-F'(0)}{3(1-\xi^3)} \right]^{1/3} \quad (24)$$

which agrees well with the results of Smith and Colton for large Schmidt numbers. The average mass-transfer rate for the same case is given by

$$\overline{St}\, Sc^{2/3}\, Re^{1/2} = \frac{1}{\Gamma(4/3)} \left[\frac{-F'(0)}{3} \right]^{1/3} \quad (25)$$

which for Bödewadt flow is 0.76114. Smith and Colton found a value of 0.768 by numerical solution of the complete partial differential equation of convection diffusion.

The experimental results of Colton and Smith (1972) show in some cases that the mass-transfer rate does not go to zero at the center of the disk and thus resemble our Figure 6. However, a quantitative comparison suggests that this effect is due to natural convection rather than to the elliptic region 2 where radial diffusion is important.

We have mentioned previously that the treatment presented here may be extended to higher order terms in the expansion for Nusselt number. The diffusion-layer solution may be extended to include the second term in the radial velocity expansion [Equation (3)]. The central and leading-edge regions must then match with this solution. This analysis is not treated here.

ACKNOWLEDGMENT

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NOTATION

- a = numerical constant in Equation (19)
- c = concentration, moles/l
- c_0 = concentration at surface, moles/l
- c_∞ = bulk concentration, moles/l
- D = diffusion coefficient, cm²/s
- f = surface concentration function for uniform wall flux condition
- F = dimensionless radial velocity function
- L = separation distance between stationary plane and rotating disk, cm
- Nu = Nusselt number
- Nu_c = composite Nusselt number
- $\overline{Nu}$ = area-averaged Nusselt number
- Pe = Péclet number, $-\frac{1}{2}F'(0)\, Sc\, Re^{3/2}$
- r = radial coordinate, cm
- R = radius of mass-transfer disk, cm
- Re = Reynolds number, $R^2\Omega/\nu$
- S = dimensionless stretched radial coordinate in central region $r(4\, Pe/3)^{1/3}/R$
- Sc = Schmidt number, ν/D
- St = Stanton number
- $\overline{St}$ = area-averaged Stanton number
- v_r = radial velocity component, cm/s
- x = integration variable in Eqution (5)
- X = dimensionless stretched radial coordinate in leading-edge region, $(1-\xi)\sqrt{Pe}$
- y = axial coordinate, cm
- Y = dimensionless stretched axial coordinate in central region, $y(4Pe/3)^{1/3}/R$
- $\overline{Y}$ = dimensionless stretched axial coordinate in leading-edge region, $y\sqrt{Pe}/R$
- Z = dimensionless separation distance between stationary plane and rotating disk, $L\sqrt{\Omega/\nu}$

Greek Letters

- β = radial velocity derivation at the plane, $-\partial v_r/\partial y$ at $y = 0$, s⁻¹
- γ = ratio of angular velocity of fluid in central core to angular velocity of rotating disk at large Z, ω/Ω
- $\Gamma(2/3)$ = the gamma function of 2/3, 1.35412
- $\Gamma(4/3)$ = the gamma function of 4/3, 0.89298
- ζ = dimensionless axial coordinate, $y\sqrt{\Omega/\nu}$
- η = Lighthill similarity variable
- θ = dimensionless radial coordinate, $(1-\xi^3)^{1/3}$
- Θ = dimensionless concentration, $(c-c_0)/(c_\infty - c_0)$
- $\overline{\Theta}_2$ = dimensionless stretched concentration for region 2, $\Theta_2\Gamma(4/3)\,(8Pe/3)^{1/3}$
- ν = kinematic viscosity, cm²/s
- ξ = dimensionless radial coordinate, r/R
- ω = angular velocity of fluid in central core at large Z, rad/s
- Ω = angular velocity of rotating disk, rad/s

Subscripts

- 1 = diffusion layer, region 1
- 2 = central region 2
- 3 = leading-edge region 3

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